

# On a foliation given by the Hecke eigenform

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## Abstract

Let  $\mathcal{F}$  be a foliation of the modular curve, given by the vertical trajectories of the Hecke eigenform. It is shown that  $\mathcal{F}$  is either a Strebel or a pseudo-Anosov foliation or else can be reduced to the above foliations. An application of the result is discussed.

*Key words and phrases:* Hecke eigenform, singular foliation

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## Introduction

Let  $N > 0$  be an integer and  $\Gamma_0(N)$  the Hecke subgroup of  $SL_2(\mathbb{Z})$ . Consider the Lobachevsky plane  $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$  and its boundary  $\partial\mathbb{H} = \mathbb{Q} \cup \{\infty\}$  made of the cusp points. Let  $S_2(\Gamma_0(N))$  be a collection of the meromorphic functions,  $f$ , on  $\mathbb{H}^* = \mathbb{H} \cup \partial\mathbb{H}$ , which vanish at the cusps and such that  $f(\alpha z) = (cz + d)^{-2}f(z)$  for all  $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $\Gamma_0(N)$  and all  $z$  in  $\mathbb{H}^*$ . Recall that the Riemann surface  $X_0(N) := \mathbb{H}^*/\Gamma_0(N)$  is called a *modular curve*. We shall routinely identify the (complex) linear spaces  $S_2(\Gamma_0(N))$  and  $\Omega_{hol}(X_0(N))$  via the formula  $f(z) \mapsto \omega = f(z)dz$ . (The formula assigns to every cusp form a holomorphic differential on the modular curve.)

Let  $f(z) = \sum c(m)q^m$  be the Fourier series of the cusp form  $f$  at the infinity. Recall that for every  $n \in \mathbb{N}$ , the Hecke operator  $T_n$  acts on  $S_2(\Gamma_0(N))$  via the formula  $T_n f = \sum \gamma(m)q^m$ , where  $\gamma(m) = \sum_{a|gcd(m,n)} ac_{mn/a^2}$ . It is well known, that  $T_n$  is a self-adjoint linear operator on  $S_2(\Gamma_0(N))$  and the Hecke operators commute with each other as  $n$  runs through  $\mathbb{N}$ . The corresponding commutative algebra will be denoted by  $\mathbb{T}_{\mathbb{Z}} := \mathbb{Z}[T_1, T_2, \dots]$ . The common eigenvector  $f \in S_2(\Gamma_0(N))$  of all  $T_n \in \mathbb{T}_{\mathbb{Z}}$  is referred to as a *Hecke eigenform*.

It is easy to see, that every  $f \in S_2(\Gamma_0(N))$  defines a singular foliation,  $\mathcal{F}$ , on  $X_0(N)$ , which is given by the vertical trajectories  $\operatorname{Re} \omega = 0$  of the holomorphic differential  $\omega = f(z)dz$ . The  $\mathcal{F}$  is a *measured* foliation [9], since the form  $\phi = \operatorname{Re} \omega$  is closed. It seems that the following interesting question has not been covered in the literature so far.

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**Main problem.** Find the topological type of the foliation  $\mathcal{F}$  defined by the Hecke eigenform  $f \in S_2(\Gamma_0(N))$ .

The aim of our note is to answer the question. Recall that the measured foliation  $\mathcal{F}$  is called *Strebel*, if all but a finite number of its leaves are closed [8]. The  $\mathcal{F}$  is called a *pseudo-Anosov* foliation, if it is the invariant foliation of an infinite order automorphism of the surface [9]. We shall call  $\mathcal{F}$  *reducible*, if  $X_0(N)$  dissects along the closed leaves of  $\mathcal{F}$  into a pair of surfaces (with the boundary), which carry a Strebel and a pseudo-Anosov foliation. For brevity, let us call the foliation defined by the real part of a Hecke eigenform a *Hecke* foliation. A summary of our results can be expressed as follows.

**Theorem 1** For every modular curve  $X_0(N)$ , the Hecke foliation is topologically conjugate to either a Strebel or a pseudo-Anosov foliation or else to a reducible foliation.

The note is organized as follows. The preliminary facts are introduced in section 1. In section 2 we prove theorem 1. In section 3 an application of our result is discussed.

## 1 Measured foliations and their jacobians

**A. Foliations.** By a  $p$ -dimensional, class  $C^r$  foliation of an  $m$ -dimensional manifold  $M$  one understands a decomposition of  $M$  into a union of disjoint connected subsets  $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ , called the *leaves* of the foliation. The leaves must satisfy the following property: Every point in  $M$  has a neighborhood  $U$  and a system of local class  $C^r$  coordinates  $x = (x^1, \dots, x^m) : U \rightarrow \mathbb{R}^m$  such that for each leaf  $\mathcal{L}_\alpha$ , the components of  $U \cap \mathcal{L}_\alpha$  are described by the equations  $x^{p+1} = \text{Const}, \dots, x^m = \text{Const}$ . Such a foliation is denoted by  $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ . The number  $q = m - p$  is called a *codimension* of the foliation  $\mathcal{F}$ , see [7] p.370. The codimension  $q$   $C^r$ -foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are said to be  $C^s$ -conjugate ( $0 \leq s \leq r$ ) if there exists a diffeomorphism of  $M$ , of class  $C^s$ , which maps the leaves of  $\mathcal{F}_0$  onto the leaves of  $\mathcal{F}_1$ . If  $s = 0$ ,  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are *topologically conjugate*, *ibid.*, p.388.

**B. Singular foliations.** The foliation  $\mathcal{F}$  is called *singular* if the codimension  $q$  of the foliation depends on the leaf. We further assume that  $q$  is constant for all but a *finite* number of leaves. Such a set of the exceptional leaves will be denoted by *Sing*  $\mathcal{F} := \{\mathcal{L}_\alpha\}_{\alpha \in F}$ , where  $|F| < \infty$ . Note that in the case  $F$  is an empty set, one gets the usual definition of a (non-singular) foliation. A quick example of the singular foliations is given by the trajectories of a non-trivial differential form on the manifold  $M$ , which vanish in a finite number of the points of  $M$ . It is easy to see that the set of zeroes of such a form corresponds to the exceptional leaves of the foliation.

**C. Measured foliations.** Roughly, the measured foliation is a singular codimension 1  $C^r$ -foliation, induced by the trajectories of a closed differential form

$\phi$  on a two-dimensional manifold (surface),  $X$ . Namely, a *measured foliation*,  $\mathcal{F}$ , on a surface  $X$  is a partition of  $X$  into the singular points  $x_1, \dots, x_n$  of order  $k_1, \dots, k_n$  and regular leaves (1-dimensional submanifolds). On each open cover  $U_i$  of  $X - \{x_1, \dots, x_n\}$  there exists a non-vanishing real-valued closed 1-form  $\phi_i$  such that: (i)  $\phi_i = \pm \phi_j$  on  $U_i \cap U_j$ ; (ii) at each  $x_i$  there exists a local chart  $(u, v) : V \rightarrow \mathbb{R}^2$  such that for  $z = u + iv$ , it holds  $\phi_i = \text{Im} (z^{\frac{k_i}{2}} dz)$  on  $V \cap U_i$  for some branch of  $z^{\frac{k_i}{2}}$ . The pair  $(U_i, \phi_i)$  is called an atlas for the measured foliation  $\mathcal{F}$ . Finally, a measure  $\mu$  is assigned to each segment  $(t_0, t) \in U_i$ , which is transverse to the leaves of  $\mathcal{F}$ , via the integral  $\mu(t_0, t) = \int_{t_0}^t \phi_i$ . The measure is invariant along the leaves of  $\mathcal{F}$ , hence the name. Note that the measured foliation can have singular points with an odd number of the saddle sections. (Those cannot be continuously oriented along the leaves and therefore cannot be given by the trajectories of a closed form.) However, when all  $k_i$  are even integers, the measured foliation can be prescribed a continuous orientation and is called *oriented*. Such foliations are given by the trajectories of a closed differential form on the surface  $X$ . In what follows, we shall work with the class of oriented measured foliations.

**D. The jacobian of a measured foliation.** Let  $\mathcal{F}$  be a measured foliation on a compact surface  $X$ . We shall assume that  $\mathcal{F}$  is an oriented foliation, i.e. given by the trajectories of a closed form  $\phi$  on  $X$ . The assumption is no restriction – each measured foliation is oriented on a surface  $\tilde{X}$ , which is a double cover of  $X$  ramified at the singular points of the half-integer index of the non-oriented foliation [6]. Let  $\{\gamma_1, \dots, \gamma_n\}$  be a basis in the relative homology group  $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$ , where  $\text{Sing } \mathcal{F}$  is the set of singular points of the foliation  $\mathcal{F}$ . It is well known (*ibid.*), that  $n = 2g + m - 1$ , where  $g$  is the genus of  $X$  and  $m = |\text{Sing } (\mathcal{F})|$ . The periods of  $\phi$  in the above basis we shall write as:  $\lambda_i = \int_{\gamma_i} \phi$ . It is known that the reals  $\lambda_i$  are coordinates of the foliation  $\mathcal{F}$  in the space of all measured foliations on the surface  $X$  (with a given set of the singular points) [5]. By a *jacobian*,  $Jac(\mathcal{F})$ , of the measured foliation  $\mathcal{F}$ , we understand a  $\mathbb{Z}$ -module  $\mathbf{m} = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n$  regarded as a subset of the real line  $\mathbb{R}$ .

**E. Properties of the jacobian.** An importance of the jacobians stems from the fact that although the periods  $\lambda_i$  depend on the basis in  $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$ , the jacobian does not. Moreover, up to a scalar multiple, the jacobian is an invariant of the topological conjugacy of the foliation  $\mathcal{F}$ . We shall formalize the observations in the following two lemmas.

**Lemma 1** *The  $\mathbb{Z}$ -module  $\mathbf{m}$  is independent of the choice of a basis in the homology group  $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$  and depends solely on the foliation  $\mathcal{F}$ .*

*Proof.* Indeed, let  $A = (a_{ij}) \in GL_n(\mathbb{Z})$  and let  $\gamma'_i = \sum_{j=1}^n a_{ij} \gamma_j$  be a new basis in  $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$ . Then using the integration rules:  $\lambda'_i = \int_{\gamma'_i} \phi = \int_{\sum_{j=1}^n a_{ij} \gamma_j} \phi = \sum_{j=1}^n \int_{\gamma_j} \phi = \sum_{j=1}^n a_{ij} \lambda_j$ . To prove that  $\mathbf{m} = \mathbf{m}'$ , consider the following equations:  $\mathbf{m}' = \sum_{i=1}^n \mathbb{Z}\lambda'_i = \sum_{i=1}^n \mathbb{Z} \sum_{j=1}^n a_{ij} \lambda_j = \sum_{j=1}^n (\sum_{i=1}^n a_{ij} \mathbb{Z}) \lambda_j \subseteq$

**m.** Let  $A^{-1} = (b_{ij}) \in GL_n(\mathbb{Z})$  be an inverse to the matrix  $A$ . Then  $\lambda_i = \sum_{j=1}^n b_{ij} \lambda'_j$  and  $\mathbf{m} = \sum_{i=1}^n \mathbb{Z} \lambda_i = \sum_{i=1}^n \mathbb{Z} \sum_{j=1}^n b_{ij} \lambda'_j = \sum_{j=1}^n (\sum_{i=1}^n b_{ij} \mathbb{Z}) \lambda'_j \subseteq \mathbf{m}'$ . Since both  $\mathbf{m}' \subseteq \mathbf{m}$  and  $\mathbf{m} \subseteq \mathbf{m}'$ , we conclude that  $\mathbf{m}' = \mathbf{m}$ . Lemma 1 follows.  $\square$

Recall that the measured foliations  $\mathcal{F}$  and  $\mathcal{F}'$  are topologically conjugate, if there exists an automorphism  $h$  of the surface  $X$ , which sends the leaves of the foliation  $\mathcal{F}$  to the leaves of the foliation  $\mathcal{F}'$ . Note that the conjugacy deals with the topological foliations (i.e. the projective classes of the measured foliations [9]) and does not preserve the transversal measure of the leaves.

**Lemma 2** *The measured foliations  $\mathcal{F}, \mathcal{F}'$  on the surface  $X$  are topologically conjugate if and only if  $Jac(\mathcal{F}') = \mu Jac(\mathcal{F})$ , where  $\mu > 0$  is a real number.*

*Proof.* Let  $h : X \rightarrow X$  be an automorphism of the surface  $X$ . Denote by  $h_*$  its action on  $H_1(X, Sing(\mathcal{F}); \mathbb{Z})$  and by  $h^*$  on  $H^1(X; \mathbb{R})$  connected by the formula:  $\int_{h_*(\gamma)} \phi = \int_{\gamma} h^*(\phi)$ ,  $\forall \gamma \in H_1(X, Sing(\mathcal{F}); \mathbb{Z})$ ,  $\forall \phi \in H^1(X; \mathbb{R})$ . Let  $\phi, \phi' \in H^1(X; \mathbb{R})$  be the closed forms, whose trajectories define the foliations  $\mathcal{F}$  and  $\mathcal{F}'$ , respectively. Since  $\mathcal{F}, \mathcal{F}'$  are topologically conjugate,  $\phi' = \mu h^*(\phi)$  for a  $\mu > 0$ .

Let  $Jac(\mathcal{F}) = \mathbb{Z} \lambda_1 + \dots + \mathbb{Z} \lambda_n$  and  $Jac(\mathcal{F}') = \mathbb{Z} \lambda'_1 + \dots + \mathbb{Z} \lambda'_n$ . Then:  $\lambda'_i = \int_{\gamma_i} \phi' = \mu \int_{\gamma_i} h^*(\phi) = \mu \int_{h_*(\gamma_i)} \phi$ ,  $1 \leq i \leq n$ . By lemma 1, it holds:  $Jac(\mathcal{F}) = \sum_{i=1}^n \mathbb{Z} \int_{\gamma_i} \phi = \sum_{i=1}^n \mathbb{Z} \int_{h_*(\gamma_i)} \phi$ . Therefore:  $Jac(\mathcal{F}') = \sum_{i=1}^n \mathbb{Z} \int_{\gamma_i} \phi' = \mu \sum_{i=1}^n \mathbb{Z} \int_{h_*(\gamma_i)} \phi = \mu Jac(\mathcal{F})$ . The converse is true as well and can be proved using a construction of the zippered rectangles over the interval exchange map, see Veech [10]. Lemma 2 follows.  $\square$

**F. Foliations given by the cusp forms.** It follows from the Hubbard-Masur main theorem [6], that the space  $S_2(\Gamma_0(N))$  is isomorphic to the space  $\Phi_{X_0(N)}$  of all measured foliations generated by the real part of the cusp forms. In particular,  $\dim_{\mathbb{R}}(S_2(\Gamma_0(N))) = \dim(\Phi_{X_0(N)}) = 2g$ , where  $g$  is the genus of  $X_0(N)$ . But  $\dim(\Phi_{X_0(N)}) = rank(Jac(\mathcal{F})) = 2g + m - 1$ . Thus, we conclude that  $m = 1$ , where  $m$  is the number of the singular points of the foliation  $\mathcal{F}$ . (The reader must not be confused comparing this result with the fact that the zeroes of the holomorphic form  $\omega = f(z)dz$  are bijective with the cusps of  $X_0(N)$ , whose number can exceed one. In the case of more than one cusp, the measured foliation with the singular saddle points, situated in the cusps, is measure equivalent to the foliation with the unique singular point via a homotopy along the saddle connections between the cusps.)

Recall that there exists a natural involution  $i$  on the space  $S_2(\Gamma_0(N))$  defined by the formula  $f(z) \mapsto f^*(z)$ , where  $f(z) = \sum c_n q^n$  and  $f^*(z) = \sum \bar{c}_n q^n$ . A subspace,  $S_2^{\mathbb{R}}(\Gamma_0(N))$ , fixed by the involution, consists of the cusp forms, whose Fourier coefficients are the real numbers. Clearly,  $\dim_{\mathbb{R}}(S_2^{\mathbb{R}}(\Gamma_0(N))) = g$ . The  $i$  induces an involution,  $i_{\Phi}$ , on the space  $\Phi_{X_0(N)}$ , which (in a proper system of the coordinates) acts by the formula  $(\lambda_1, \dots, \lambda_g, \lambda'_1, \dots, \lambda'_g) \mapsto (\lambda'_1, \dots, \lambda'_g, \lambda_1, \dots, \lambda_g)$ . It is easy to see, that the  $i_{\Phi}$ -invariant subspace,  $\Phi_{X_0(N)}^{\mathbb{R}}$ , consists of the mea-

sured foliations  $(\lambda_1, \dots, \lambda_g, \lambda_1, \dots, \lambda_g)$ . Thus,  $Jac(\mathcal{F}) = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$  for  $\forall \mathcal{F} \in \Phi_{X_0(N)}^{\mathbb{R}}$ .

**G. An elementary lemma.** Finally, let us recall the following statement from linear algebra, which will be used for the proof of theorem 1.

**Lemma 3** *Let  $T \in M_n(\mathbb{Z})$  be a linear endomorphism of the vector space  $\mathbb{R}^n$ , where  $M_n(\mathbb{Z})$  is the set of the  $n \times n$  matrices over  $\mathbb{Z}$ . If  $Tx = \lambda x$ , where  $x \in \mathbb{R}^n$  is an eigenvector and  $\lambda \in K$  is an eigenvalue of  $T$ , then  $x$  can be scaled so that  $x \in K^n$ , where  $K = \mathbb{Q}(\lambda)$  is a subfield of  $\mathbb{R}$  generated by  $\lambda$ .*

*Proof.* Let  $T = (t_{ij}) \in M_n(\mathbb{Z})$ . Then the eigenvalue equation  $Tx = \lambda x$  unfolds as:

$$\begin{cases} t_{11}x_1 + t_{12}x_2 + \dots + t_{1n}x_n &= \lambda x_1 \\ t_{21}x_1 + t_{22}x_2 + \dots + t_{2n}x_n &= \lambda x_2 \\ &\vdots \\ t_{n1}x_1 + t_{n2}x_2 + \dots + t_{nn}x_n &= \lambda x_n \end{cases} \quad (1)$$

Let us scale the vector  $x \in \mathbb{R}^n$  so that  $x_1 = 1$ . One can rewrite the above system of equations as:

$$\begin{cases} t_{12}x_2 + \dots + t_{1n}x_n &= \lambda - t_{11} \\ (t_{22} - \lambda)x_2 + \dots + t_{2n}x_n &= -t_{21} \\ &\vdots \\ t_{n2}x_2 + \dots + (t_{nn} - \lambda)x_n &= -t_{n1} \end{cases} \quad (2)$$

To solve the above system for the variables  $x_2, \dots, x_n$ , let us recall that the rank  $(T - \lambda I) = n - 1$  and therefore one can cancel any line in the system so as to obtain a unique solution. Let it be the first line. Then:

$$\begin{pmatrix} t_{22} - \lambda & \dots & t_{2n} \\ \vdots & & \vdots \\ t_{n2} & \dots & t_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -t_{21} \\ \vdots \\ -t_{n1} \end{pmatrix} \quad (3)$$

is a matrix form for the above linear equations. Note that

$$\Delta = \det \begin{pmatrix} t_{22} - \lambda & \dots & t_{2n} \\ \vdots & & \vdots \\ t_{n2} & \dots & t_{nn} - \lambda \end{pmatrix} \in K, \quad (4)$$

since the entries of the matrix belong to the field  $K$ . By the Kronecker formulas,  $x_j = \Delta_j / \Delta$ , where  $\Delta_j$  are the determinants of the matrices whose  $j$ -th column is replaced by  $(-t_{21}, \dots, -t_{n1})^T$ . Since  $\Delta_j \in K$  as well, we conclude that  $x_j \in K$  for all  $2 \leq j \leq n$ . Lemma 3 follows.  $\square$

## 2 Proof of theorem 1

Let  $f \in S_2(\Gamma_0(N))$  be a (normalized) Hecke eigenform, such that  $f(z) = \sum_{n=1}^{\infty} c_n(f)q^n$  its Fourier series. We shall denote by  $K_f = \mathbb{Q}(\{c_n(f)\})$  the

algebraic number field generated by the Fourier coefficients of  $f$ . Let  $g$  be the genus of the modular curve  $X_0(N)$ . It is well known that  $1 \leq \deg(K_f | \mathbb{Q}) \leq g$  and  $K_f$  is a totally real field, see e.g. [3], p. 25. Let  $\mathcal{F}$  be the Hecke foliation of  $f$ . The following lemma will be critical.

**Lemma 4**  *$Jac(\mathcal{F})$  is a  $\mathbb{Z}$ -module in the field  $K_f$ .*

*Proof.* Let  $\phi = \text{Re}(fdz)$ , where  $f$  is the Hecke eigenform. By the definition,  $Jac(\mathcal{F}) = \int_{H_1(X_0(N), \text{Sing } \mathcal{F}; \mathbb{Z})} \phi = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$ ,  $\lambda_i \in \mathbb{R}$ . Let us study the action of the Hecke operators on the  $Jac(\mathcal{F})$ . Since  $f$  is an eigenform,  $T_n f = c_n f$ ,  $\forall T_n \in \mathbb{T}_{\mathbb{Z}}$ . By virtue of the isomorphism  $S_2(\Gamma_0(N)) \cong \Omega_{hol}(X_0(N))$ , one gets  $T_n \omega = c_n \omega$ , where  $c_n \in K_f$ . Let us evaluate the real parts of the last equation as follows:  $\text{Re}(T_n \omega) = T_n(\text{Re } \omega) = \text{Re}(c_n \omega) = c_n(\text{Re } \omega)$ . (Note that the equality  $\text{Re}(c_n \omega) = c_n(\text{Re } \omega)$  involves the fact that  $c_n$  is a real number.) Therefore, one concludes that  $T_n \phi = c_n \phi$ ,  $\forall T_n \in \mathbb{T}_{\mathbb{Z}}$  and  $c_n \in K_f$ . Therefore, the action of the Hecke operator  $T_n$  on the  $Jac(\mathcal{F})$  can be written as:

$$T_n(Jac(\mathcal{F})) = \int_{H_1} T_n \phi = \int_{H_1} c_n \phi = c_n Jac(\mathcal{F}), \quad c_n \in K_f, \quad (5)$$

where  $H_1 = H_1(X_0(N), \text{Sing } \mathcal{F}; \mathbb{Z})$  is the relative homology group. Thus,  $T_n$  acts multiplicatively on the  $Jac(\mathcal{F})$  by the real numbers  $c_n$ .

Consider the  $\mathbb{Z}$ -module  $Jac(\mathcal{F}) = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$ , where  $\lambda_i \in \mathbb{R}$ . It is easy to see, that the Hecke operator  $T_n$  is an endomorphism of the vector space  $\{\lambda \in \mathbb{R}^g \mid \lambda = (\lambda_1, \dots, \lambda_g)\}$ . By the virtue of (5), the vector  $\lambda \in \mathbb{R}^g$  is an eigenvector of the linear operator  $T_n \in M_g(\mathbb{Z})$ . In other words,  $T_n \lambda = c_n \lambda$ ,  $c_n \in K_f$ . In view of the lemma 3, one can scale the vector  $\lambda$  so that  $\lambda_i \in K_f$ .  $\square$

**Lemma 5**  *$\text{rank}(Jac(\mathcal{F})) = \deg(K_f | \mathbb{Q})$ .*

*Proof.* Let  $I_f = \{T \in \mathbb{T}_{\mathbb{Z}} : Tf = 0\}$  and  $R = \mathbb{T}_{\mathbb{Z}}/I_f$ . It is well known that  $R$  is an order in the number field  $K_f$ , i.e. a subring of the ring of integers of  $K_f$  containing 1. Recall that a coefficient ring of the lattice  $\Lambda$  in  $K_f$  is defined as the set  $\alpha \in K_f$ , such that  $\alpha\Lambda \subseteq \Lambda$ . The coefficient ring is always an order in  $K_f$ . It is verified directly, that  $R$  is the coefficient ring of the lattice  $\Lambda = Jac(\mathcal{F})$ . Moreover, the  $Jac(\mathcal{F})$  can be scaled so that  $Jac(\mathcal{F}) \subseteq R$ , see [2], p.88. Therefore,  $Jac(\mathcal{F})$  and  $R$  have the same number of the generators (a rank), see *ibid.* But  $\text{rank}(R) = \deg(K_f | \mathbb{Q})$ , see e.g. [4], p.234. Thus,  $\text{rank}(Jac(\mathcal{F})) = \deg(K_f | \mathbb{Q})$ .  $\square$

**Lemma 6** *The Hecke foliation is topologically conjugate to:*

- (i) a *Strebel foliation*, if  $\deg(K_f | \mathbb{Q}) = 1$ ;
- (ii) a *pseudo-Anosov foliation*, if  $\deg(K_f | \mathbb{Q}) = g$ ;
- (iii) a *reducible foliation* for otherwise.

*Proof.* We shall use the standard dictionary between the measured foliations and (the generators  $\lambda_i > 0$  of) their jacobians. Namely, the  $\lambda_i$  measure the length

of the intervals in an interval exchange map induced by the given foliation  $\mathcal{F}$  on a closed transversal to the foliation [10]. Conversely, given  $\lambda_i$  one recovers a measured foliation  $\mathcal{F}$  by a zippering of the rectangles based at  $\lambda_i$ , see *ibid.* According to the dictionary, the following three cases must be considered.

(i) Let  $\deg(K_f | \mathbb{Q}) = 1$  (the minimal possible). By lemma 4,  $\text{rank}(Jac(\mathcal{F})) = 1$  and therefore, after a scaling, all  $\lambda_i$  are rational. Since  $\lambda_i$  are linearly dependent over  $\mathbb{Q}$ , the orbit of each point (except the endpoints) under the interval exchange map is periodic, see the above cited paper of W. Veech. In this case, the zippered rectangles produce a measured foliation all of whose leaves, but a finite number, are closed. The foliation, by historical reasons, is called the Strebel foliation [8].

(ii) Let  $\deg(K_f | \mathbb{Q}) = g$  (the maximal possible). In this case  $\lambda_i$  are linearly independent over  $\mathbb{Q}$ . Therefore, the foliation  $\mathcal{F}$  is minimal, i.e. each leaf of  $\mathcal{F}$  (except the singular leaf) is dense on the surface  $X_0(N)$ . Let us show, that  $\mathcal{F}$  is the invariant foliation of a pseudo-Anosov automorphism of  $X_0(N)$ .

Indeed, recall that  $Jac(\mathcal{F})$  is a full module in the number field  $K_f$ . Let  $\lambda_i > 0$  and  $\theta = (\theta_1, \dots, \theta_{g-1})$ , where  $\theta_i = \lambda_{i-1}/\lambda_1$ . Recall that, up to a scalar multiple, a (generic) vector  $(1, \theta) \in \mathbb{R}^g$  is the limit of a convergent Jacobi-Perron continued fraction [1]:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k \rightarrow \infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \quad (6)$$

where  $b_i = (b_1^{(i)}, \dots, b_{g-1}^{(i)})^T$  is a vector of the non-negative integers,  $I$  the unit matrix and  $\mathbb{I} = (0, \dots, 0, 1)^T$ . Moreover, the periodic continued fractions converge to the vectors with  $\theta_i \in K_f$ . For simplicity, we assume our continued fraction to be purely periodic and we let  $A \in GL_g(\mathbb{Z})$  be the matrix of the minimal period of the Jacobi-Perron continued fraction. (Such a matrix is a finite product of the elementary matrices  $B_i$ , which appear in the formula (6).) Clearly,  $A\lambda = \lambda_A \lambda$ , where  $\lambda_A$  is the Perron-Frobenius eigenvalue of  $A$ . Note that  $\lambda_A \in K_f$  and  $\lambda_A$  is a unit of the order  $R = \mathbb{T}_{\mathbb{Z}}/I_f$ , which can be called a *Hecke unit*. It is easy to see, that the Hecke unit is an invariant of the  $Jac(\mathcal{F})$  in the sense that it does not depend on the choice of the generators of the jacobian.

Recall (§1, F), that each  $\mathcal{F} \in \Phi_{X_0(N)}^{\mathbb{R}}$  is induced by the trajectories of closed differential  $\phi \in H^1(X_0(N); \mathbb{R})$ . In the presence of the involution  $i_{\Phi}$ , the above cohomology splits into the sum  $H_{\mathbb{R}}^1 \oplus H_{i_{\mathbb{R}}}^1$ , so that the matrix  $A$  defines a linear operator on the vector space  $H_{\mathbb{R}}^1(X_0(N); \mathbb{R})$ . By the involution, the linear operator extends to the entire  $H^1(X_0(N); \mathbb{R})$ . It is easy to see, that  $A\phi = \lambda_A \phi$  for the vector  $\phi = (\lambda_1, \dots, \lambda_g)$ . On the other hand, any linear operator on the cohomology group of a surface is induced by an automorphism,  $\varphi$ , of the surface. Thus, if  $\mathcal{F}$  is a foliation by the trajectories of  $\phi$ , the  $\mathcal{F}$  is an invariant foliation of  $\varphi$ . The matrix  $A$  is a hyperbolic matrix, so the automorphism has an infinite order, i.e. is a pseudo-Anosov automorphism. Clearly, the Hecke unit coincides with the dilatation of  $\varphi$ , see [9] for a definition of the dilatation.

(iii) Let  $\deg (K_f | \mathbb{Q}) = n$ , where  $2 \leq n \leq g - 1$ . Recall (§1, F), that the Hecke foliation has a unique singularity, which is a saddle point with the  $(4g - 2)$ -separatrices. Since some  $\lambda_i$  are linearly dependent (over  $\mathbb{Q}$ ), in this case the method of zippered rectangles produces a foliation with the separatrix cycles, i.e the separatrices in the above mentioned saddle make loops beginning and ending in the saddle. Such cycles come in pairs, which bound a cylinder filled up with the closed leaves. Dissecting the surface  $X_0(N)$  along the separatrix cycles gives two surfaces of smaller genera, on which one can repeat the argument of the item (i) or (ii). Clearly, the foliation on the surfaces is either the Strebel or the pseudo-Anosov foliation.  $\square$

Theorem 1 follows from the lemma 6.  $\square$

### 3 An application

**A. The Anosov-Hecke eigenforms.** Let us call the eigenform  $f$  an *Anosov-Hecke eigenform*, whenever  $\deg (K_f | \mathbb{Q}) = g$ , i.e. the maximal possible. Note that such eigenforms can be viewed as an opposite to the rational eigenforms, whose Fourier coefficients are rational, i.e. the field  $K_f = \mathbb{Q}$ . It is known that the rational eigenforms are critical in the context of the Eichler-Shimura construction of a rational elliptic curve from the given rational eigenform. In the present section, it is shown how to attach a rational elliptic curve to the Anosov-Hecke eigenform using a dynamical system generated by the pseudo-Anosov automorphism of the surface  $X_0(N)$ .

**B. The Anosov  $L$ -function.** Let  $f$  be an Anosov-Hecke eigenform and  $A \in GL_g(\mathbb{Z})$  the matrix associated to  $f$ , see (ii) of the proof of lemma 6. For any power  $n$  of  $A$  consider the abelian group  $BF_n(A) := \mathbb{Z}^g / (A^n - I)\mathbb{Z}^g$ , where  $I$  is the unit matrix. The fact that  $A$  is a hyperbolic matrix implies that  $BF_n(A)$  is a finite group for any  $n \in \mathbb{N}$ . (For  $n = 1$  the group is known as the Bowen-Franks group of an automorphism  $A$  of the lattice  $\mathbb{Z}^g$ , hence our notation.) For every prime number  $p$ , denote by

$$\zeta_p(A, z) := \exp \left( \sum_{n=1}^{\infty} \frac{|BF_{p^n}(A)|}{n} z^n \right), \quad z \in \mathbb{C},$$

a local zeta function attached to the hyperbolic matrix  $A$ . The reader can recognize that  $\zeta_p(A, z)$  coincides with the Artin-Mazur zeta function of the automorphism  $A^p$ . The series is known to have positive radius of convergence to a rational function. We let

$$L(A, s) = \prod_{p \text{ prime}} \zeta_p(A, p^{-s}), \quad s \in \mathbb{C},$$

be a product of the local zeta functions over all prime numbers  $p$ . We shall call  $L(A, s)$  an *Anosov  $L$ -function* of the automorphism  $A \in GL_g(\mathbb{Z})$ . Let us conclude by the following conjecture.



**Conjecture 1** *For every Anosov-Hecke eigenform, the Anosov  $L$ -function coincides with the Hasse-Weil zeta function of a rational elliptic curve. Conversely, every rational elliptic curve can be attached an Anosov-Hecke eigenform (on some  $X_0(N)$ ), whose Anosov  $L$ -function coincides with the Hasse-Weil zeta function of the elliptic curve.*

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